B.Sc part – II(H), paper – III 26-05-2020

Q.1. State and prove Gauss's test.

Soln: Statement: → If $\frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + \frac{\lambda_n}{n^2}$, where the sequence $\{\lambda_n\}$ is bounded, then the series $\sum u_n$ converges or diverges according as $\mu > 1$ or ≤ 1.

Proof: $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

Hence, D' Alembert's ratio test fails.

Since, $\{\lambda_n\}$ is bounded, therefore, $|\lambda_n| < K$, $n = 1, 2, \ldots$

$$\therefore \lim_{n\to\infty} \left|\frac{\lambda_n}{n}\right| < \lim_{n\to\infty} \frac{K}{n} = 0$$

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Now, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \left(\mu + \frac{\lambda_n}{n}\right) = \mu.$

Hence, by Raabe's test, $\sum u_n$ converges when $\mu > 1$ and diverges when $\mu < 1$.

When $\mu = 1$, Raabe's test fails.

Now, $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{\lambda_n}{n^2}.$

$$\therefore \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n = \frac{\lambda_n \log n}{n} < K\frac{\log n}{n}$$

$$\therefore \lim_{n\to\infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] < \lim_{n\to\infty} K\frac{\log n}{n} = 0 < 1.$$

Hence, by De Morgan and Bertrand's test, $\sum u_n$ diverges when $\mu = 1$.

Proved

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Q.2. Apply Gauss's test to examine the convergence of the series.

$$\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \dots \text{ to } \infty.$$

Soln: Let the general term of the given series be denoted by u_n.

Then, $$u_n = \frac{1^2.3^2.5^2 \dots (2n-1)^2}{2^2.4^2.6^2 \dots (2n)^2}$$

And $$u_{n+1} = \frac{1^2.3^2.5^2 \dots (2n-1)^2 (2n+1)^2}{2^2.4^2.6^2 \dots (2n)^2 (2n+2)^2}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$= \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left\{1 - 2\cdot\frac{1}{2n} + 3\cdot\left(\frac{1}{2n}\right)^2 - \dots\right\}$$

By Binomial theorem

$$= 1 + \frac{1}{n} + \frac{1}{n^2}\left\{-\frac{1}{4} + \text{terms containing } \frac{1}{n^2}, \text{etc}\right\}$$

$$= 1 + \frac{1}{n} + \frac{b_n}{n^2}, \text{ where } b_n \to -\frac{1}{4} \text{ as } n \to \infty.$$

Hence, Gauss's test suggests us to accept that the given series is divergent, as here $\mu = 1$.

Hence, the Result.